Holonomy of supermanifolds

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The case of smooth manifolds

Let $E \to M$ be a vector bundle over a smooth manifold M, ∇ a connection on E.

$\gamma : [a, b] \to M$ a curve in M

$\tau_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$ the parallel transport along γ

$x \in M$, $\tau_{pt_x} = \mathrm{id}_{E_x}$, $\tau_{\gamma \star \mu} = \tau_\mu \circ \tau_\gamma$, $\tau_{\gamma^{-1}} = (\tau_\gamma)^{-1}$.

The holonomy group at the point x:

$$\mathrm{Hol}_x(\nabla) := \{\tau_\gamma \big| \gamma \text{ is a loop at } x\} \subset \mathrm{GL}(E_x) \simeq \mathrm{GL}(m, \mathbb{R}).$$

The restricted holonomy group at the point x:

$$\mathrm{Hol}^0_x(\nabla) := \{\tau_\gamma \big| \gamma \text{ is a loop at } x, \gamma \sim \mathrm{pt}_x\} \subset \mathrm{Hol}_x(\nabla).$$

Fact: $\mathrm{Hol}_x(\nabla) \subset \mathrm{GL}(E_x)$ *is a Lie subgroup,*
$\mathrm{Hol}^0_x(\nabla)$ *is the identity component of* $\mathrm{Hol}_x(\nabla)$*.*

The holonomy algebra at the point x:

$$\mathfrak{hol}_x(\nabla) := \mathrm{LA}\,\mathrm{Hol}_x(\nabla) = \mathrm{LA}\,\mathrm{Hol}^0_x(\nabla) \subset \mathfrak{gl}(E_x) \simeq \mathfrak{gl}(m, \mathbb{R}).$$

Theorem. (Ambrose, Singer, 1952)

$$\mathfrak{hol}_x(\nabla) = \{(\tau_\gamma)^{-1}\circ R_{\gamma(b)}(\tau_\gamma(X), \tau_\gamma(Y))\circ\tau_\gamma \,\big|\, \gamma(a) = x,\ X, Y \in T_xM\}.$$

The fundamental principle:

$\{$ parallel sections $X \in \Gamma(E)\} \longleftrightarrow \{X_x \in E_x \,|\, Hol_x X_x = X_x\}$

($X \in \Gamma(E)$ is parallel if $\nabla X = 0$, or for any $\gamma : [a, b] \to M$, $\tau_g ammaX_{\gamma(a)} = X_{\gamma(a)}$)

Holonomy of Riemannian manifolds

$(M,g), \quad E = TM, \quad \nabla = \nabla^g, \quad \mathrm{Hol}(\nabla) \subset O(n), \, \mathfrak{hol}(\nabla) \subset \mathfrak{so}(n)$

Consider two Riemannian manifolds (M,g), (N,h), then $(M \times N, g+h)$ is also a Riemannian manifold and

$$\mathrm{Hol}(M \times N) = \mathrm{Hol}(M) \times \mathrm{Hol}(N).$$

Conversely:

Theorem (De Rham) *If (M,g) is complete and simply connected, then*

$$M = N_0 \times N_1 \times \cdots \times N_r,$$

$$\mathrm{Hol}(M) = \{\mathrm{id}\} \times \mathrm{Hol}(N_1) \times \cdots \times \mathrm{Hol}(N_r),$$

$\mathrm{Hol}(N_i)$ *are irreducible.*

In general exists local decomposition of (M,g).

If (M, g) is an indecomposable simply connected symmetric Riemannian space, then

$$M = G/H,$$

where G is the group of transvections,

then Hol coincides with the isotropy representation of H.

Simply connected symmetric Riemannian spaces are classified, hence all possible Hol are known.

Connected irreducible holonomy groups of non-locally symmetric Riemannian manifolds (M. Berger 1953):

$$SO(n),\ U\left(\frac{n}{2}\right),\ SU\left(\frac{n}{2}\right),\ Sp\left(\frac{n}{4}\right),\ Sp\left(\frac{n}{4}\right)\cdot Sp(1),$$

$$\mathrm{Spin}(7)\subset SO(8),\ G_2\subset SO(7).$$

Special geometries:

$SO(n)$: "general" Riemannian manifolds;

$U(\frac{n}{2})$: Kählerian manifolds;

$SU(\frac{n}{2})$: Calabi-Yau manifolds or special Kählerian manifolds, $\mathrm{Ric} = 0$, parallel spinors;

$Sp(\frac{n}{4})$: hyper-Kählerian manifolds, $\mathrm{Ric} = 0$, parallel spinors;

$Sp(\frac{n}{4}) \cdot Sp(1)$: quaternionic-Kählerian manifolds, Einstein;

$\mathrm{Spin}(7)$: 8-dimensional manifolds with a parallel 4-form, $\mathrm{Ric} = 0$, parallel spinors;

G_2: 7-dimensional manifolds with a parallel 3-form, $\mathrm{Ric} = 0$, parallel spinors.

Supermanifolds

Let $\mathcal{E}$ be a locally free sheaf of supermodules over $\mathcal{O}_{\mathcal{M}}$ of rank $p|q$.

$x \in M$ consider the fiber at x: $\mathcal{E}_x := \mathcal{E}(U)/(\mathcal{O}_{\mathcal{M}}(U))_x\mathcal{E}(U)$,

where $x \in U$ and $(\mathcal{O}_{\mathcal{M}}(U))_x \subset \mathcal{O}_{\mathcal{M}}(U)$ are functions vanishing at x.

For $X \in \mathcal{E}(U)$ consider the value $X_x \in \mathcal{E}_x$

Example. $\mathcal{E} = \mathcal{T}_{\mathcal{M}} \Rightarrow (\mathcal{T}_{\mathcal{M}})_x = T_x\mathcal{M}$ and $(T_x\mathcal{M})_{\bar{0}} = T_xM$

Let $\mathcal{E}$ be a locally free sheaf of supermodules over $\mathcal{O}_{\mathcal{M}}$ of rank $p|q$.

Consider the vector bundle $E = \cup_{x \in M} \mathcal{E}_x \to M$.

We get the projection $\sim$: $\mathcal{E}(U) \to \Gamma(U, E), \quad X \mapsto \tilde{X}, \quad \tilde{X}_x = X_x$

Let (e_A) $A = 1, ..., p+q$ be a basis of $\mathcal{E}(U)$

$X \in \mathcal{E}(U) \Rightarrow X = X^A e_A \; (X^A \in \mathcal{O}_{\mathcal{M}}(U)) \Rightarrow \tilde{X} = \tilde{X}^A \tilde{e}_A$

$X \in \mathcal{E}(U)$ is not defined by its values!

Connection on $\mathcal{E}$: $\quad \nabla : \mathcal{T}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathcal{E} \to \mathcal{E} \quad |\nabla_\xi X| = |\xi| + |X|$,

$$\nabla_{f\xi} X = f \nabla_\xi X \quad \text{and} \quad \nabla_\xi f X = (\xi f) X + (-1)^{|\xi||f|} f \nabla_\xi X$$

Locally: $\nabla_{\partial_a} e_B = \Gamma^A_{aB} e_A, \quad \Gamma^A_{aB} \in \mathcal{O}_{\mathcal{M}}(U)$

$\tilde{\nabla} = (\nabla|_{\Gamma(TM) \otimes \Gamma(E)})^\sim : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$ is a connection on E

$\tilde{\Gamma}^A_{iB}$ are Cristoffel symbols of $\tilde{\nabla}$

$\gamma : [a, b] \subset \mathbb{R} \to M \quad \tau_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$ the parallel displac. along γ (defined by $\tilde{\nabla}$).

$\tau_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ is an isomorphism of vector superspaces.

Problem: Define holonomy of ∇ (it must give information about all parallel sections of $\mathcal{E}$!)

Parallel sections

$X \in \mathcal{E}(M)$ is called parallel if $\nabla X = 0$

$\nabla X = 0 \Rightarrow \tilde{\nabla}\tilde{X} = 0 \quad (\not\Leftarrow!!!)$

Locally:

$$\nabla X = 0 \Leftrightarrow \begin{cases} \partial_i X^A + X^B \Gamma^A_{iB} = 0, \\ \partial_\gamma X^A + (-1)^{|X^B|} X^B \Gamma^A_{\gamma B} \quad = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\partial_{\gamma_r} \dots \partial_{\gamma_1}(\partial_i X^A + X^B \Gamma^A_{iB}))^\sim = 0, & (*) \\ (\partial_{\gamma_r} \dots \partial_{\gamma_1}(\partial_\gamma X^A + (-1)^{|X^B|} X^B \Gamma^A_{\gamma B}))^\sim = 0 & (**) \end{cases} \quad r = 0, \dots, m$$

$$\tilde{\nabla}\tilde{X} = 0 \Leftrightarrow \partial_i \tilde{X}^A + \tilde{X}^B \tilde{\Gamma}^A_{iB} = 0$$

Proposition. A parallel section $X \in \mathcal{E}(M)$ is uniquely defined by its value at any point $x \in M$.

Proof. $\nabla X = 0 \Rightarrow \tilde{\nabla}\tilde{X} = 0$; $\tilde{X}_x = X_x$ uniquely determine $\tilde{X}$, i.e. we know the functions $\tilde{X}^A$.

Further, use $(**)$: $X^A_\gamma = -\tilde{X}^B \tilde{\Gamma}^A_{\gamma B}$,
$X^A_{\gamma\gamma_1} = -\tilde{X}^B \Gamma^A_{\gamma B \gamma_1} + X^B_{\gamma_1} \tilde{\Gamma}^A_{\gamma B}$ ... $\Rightarrow$ we know the functions X^A. $\square$

Definition (holonomy algebra)

$\mathfrak{hol}(\nabla)_x :=$

$$\left\langle \tau_\gamma^{-1} \circ \bar{\bar{\nabla}}^r_{Y_r,\dots,Y_1} R_y(Y,Z) \circ \tau_\gamma \,\middle|\, \begin{array}{c} r \geq 0,\; Y, Z, Y_i \in T_y\mathcal{M} \\ \bar{\bar{\nabla}}\text{: connect on } \mathcal{T}_\mathcal{M}|_U \end{array} \right\rangle \subset \mathfrak{gl}(\mathcal{E}_x)$$

Note: $\mathfrak{hol}(\tilde{\nabla})_x \subset (\mathfrak{hol}(\nabla)_x)_{\bar{0}} \qquad (\neq !)$

Lie supergroup $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ is a group object in the category of supermanifolds; $\mathcal{G}$ is uniquely given by the Harish-Chandra pair $(G, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G.

Denote by $\mathrm{Hol}(\nabla)^0_x$ the connected Lie subgroup of $\mathrm{GL}((\mathcal{E}_x)_{\bar{0}}) \times \mathrm{GL}((\mathcal{E}_x)_{\bar{1}})$ corresponding to $(\mathfrak{hol}(\nabla)_x)_{\bar{0}} \subset \mathfrak{gl}((\mathcal{E}_x)_{\bar{0}}) \oplus \mathfrak{gl}((\mathcal{E}_x)_{\bar{1}}) \subset \mathfrak{gl}(\mathcal{E}_x)$;

$\mathrm{Hol}(\nabla)_x := \mathrm{Hol}(\nabla)^0_x \cdot \mathrm{Hol}(\tilde{\nabla})_x \subset \mathrm{GL}((\mathcal{E}_x)_{\bar{0}}) \times \mathrm{GL}((\mathcal{E}_x)_{\bar{1}})$.

Def. Holonomy group: $\mathcal{H}ol(\nabla)_x := (\mathrm{Hol}(\nabla)_x, \mathfrak{hol}(\nabla)_x)$;

the restricted holonomy group: $\mathcal{H}ol(\nabla)^0_x := (\mathrm{Hol}(\nabla)^0_x, \mathfrak{hol}(\nabla)_x)$.

Definition (Infinitesimal holonomy algebra).

$$\mathfrak{hol}(\nabla)_x^{inf} := \\ < \bar{\nabla}^r_{Y_r,\dots,Y_1} R_x(Y,Z) | r \geq 0,\ \ Y, Z, Y_1, \dots, Y_r \in T_x\mathcal{M} > \subset \mathfrak{hol}(\nabla)_x$$

Theorem. If $\mathcal{M}$, $\mathcal{E}$ and ∇ are analytic, then $\mathfrak{hol}(\nabla)_x = \mathfrak{hol}(\nabla)_x^{inf}$.

Theorem.

$$\{X \in \mathcal{E}(M),\ \ \nabla X = 0\} \longleftrightarrow \left\{ \begin{array}{c} X_x \in \mathcal{E}_x \text{ annihilated by } \mathfrak{hol}(\nabla)_x \\ \text{and preserved by } \mathrm{Hol}(\tilde{\nabla})_x \end{array} \right\}$$

Proof. $\longrightarrow$: $\quad \nabla X = 0 \quad \Rightarrow \quad \bar{\nabla}^r_{Y_r,\dots,Y_1} R(Y,Z)X = 0$

$\nabla X = 0 \quad \Rightarrow \quad \tilde{\nabla}\tilde{X} = 0 \quad \Rightarrow \quad \tilde{X}$ is preserved by $\mathrm{Hol}(\tilde{\nabla})_x$

$\Longrightarrow \quad \bar{\nabla}^r_{Y_r,\dots,Y_1} R_y(Y,Z) \circ \tau_\gamma X_x = 0 \quad \Rightarrow \quad X_x$ is annihilated by $\mathfrak{hol}(\nabla)_x$

$\longleftarrow$:

$\mathrm{Hol}(\tilde{\nabla})_x$ preserves $X_x \in \mathcal{E}_x$

$\Longrightarrow \ \exists X_0 \in \Gamma(E),\ \tilde{\nabla} X_0 = 0,\ (X_0)_x = X_x$

$X_0 = X_0^A \tilde{e}_A,\ X_0^A \in \mathcal{O}_M(U)$

$(**)$ defines $X^A_{\gamma\gamma_1\dots\gamma_r} \in \mathcal{O}_M(U)$ for all $\gamma < \gamma_1 < \cdots < \gamma_r$, $0 \le r \le m-1$.

We get $X^A \in \mathcal{O}_{\mathcal{M}}(U)$, consider $X = X^A e_A \in \mathcal{E}(U)$.

Claim: $\nabla X = 0$. To prove (by induction over r): X^A satisfy $(*)$ and $(**)$ for all $\gamma_1 < \cdots < \gamma_r$, $0 \le r \le m$

$$
\begin{aligned}
(\partial_{\gamma_r}\dots\partial_{\gamma_1}(\partial_i X^A + X^B \Gamma^A_{iB}))^\sim &= (\partial_{\gamma_r}\dots\partial_{\gamma_2}((-1)^{(|A|+|B|)|X^B|} R^A_{B\gamma_1 i} X^B))^\sim \\
&= (\partial_{\gamma_r}\dots\partial_{\gamma_3}((-1)^{(|A|+|B|)|X^B|} \bar{\nabla}_{\gamma_2} R^A_{B\gamma_1 i} X^B))^\sim \\
&= \cdots = ((-1)^{(|A|+|B|)|X^B|} \bar{\nabla}^{r-1}_{\gamma_r,\dots,\gamma_2} R^A_{B\gamma_1 i} X^B)^\sim = 0,
\end{aligned}
$$

this proves $(*)$

Linear connections

∇ a connection on $\mathcal{E} = \mathcal{T}_{\mathcal{M}}$,

$E = \cup_{y \in M} T_y\mathcal{M} = T\mathcal{M}, \quad E_{\bar{0}} = TM$

$\mathfrak{hol}(\nabla) \subset \mathfrak{gl}(n|m, \mathbb{R}), \qquad \mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(m, \mathbb{R})$

Theorem.

$$\left\{ \begin{array}{c} \text{Parallel tensor fields} \\ \text{of type } (p,q) \text{ on } \mathcal{M} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} A_x \in T_x^{p,q}\mathcal{M} \text{ annihilated by } \mathfrak{hol}(\nabla)_x \\ \text{and preserved by } \mathrm{Hol}(\tilde{\nabla})_x \end{array} \right\}$$

Examples of parallel structures on $(\mathcal{M}, \nabla)$

parallel structure on $\mathcal{M}$	$\mathfrak{hol}(\nabla)$ is contained in	$\mathrm{Hol}(\tilde{\nabla})$ is contained in
complex structure	$\mathfrak{gl}(k\|l, \mathbb{C})$	$\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(l, \mathbb{C})$
odd complex structure, i.e. odd automorphism J of $\mathcal{T}_\mathcal{M}$ with $J^2 = -\mathrm{id}$	$\mathfrak{q}(n, \mathbb{R})$ (queer Lie superalgebra)	$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right) \middle\| A \in \mathrm{GL}(n, \mathbb{R}) \right\}$
Riemannian supermetric, i.e. even non-degenerate supersymmetric metric	$\mathfrak{osp}(p_0, q_0\|2k)$	$\mathrm{O}(p_0, q_0) \times \mathrm{Sp}(2k, \mathbb{R})$
even non-degenerate super skew-symmetr. metric	$\mathfrak{osp}^{\mathrm{sk}}(2k\|p, q)$	$\mathrm{Sp}(2k, \mathbb{R}) \times \mathrm{O}(p, q)$
odd non-degenerate supersymmetric metric	$\mathfrak{pe}(n, \mathbb{R})$ (periplectic Lie superalgebra)	$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right) \middle\| A \in \mathrm{GL}(n, \mathbb{R}) \right\}$
odd non-degenerate super skew-symmetric metric	$\mathfrak{pe}^{sk}(n, \mathbb{R})$	$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right) \middle\| A \in \mathrm{GL}(n, \mathbb{R}) \right\}$

Riemannian supermanifolds

$(\mathcal{M}, g)$, where

$$g : \mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$$

is a symmetric even nondegenerate

g defines a pseudo-Riemannian metric $\tilde{g}$ (of signature (p, q)) on M.

On $(\mathcal{M}, g)$ exists a unique Levi-Civita connection ∇

$\mathfrak{hol}(\mathcal{M}, g) \subset \mathfrak{osp}(p, q|2k)$ and $\mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{O}(p, q) \times \mathrm{Sp}(2k, \mathbb{R})$

Special geometries of Riemannian supermanifolds and the corresponding holonomies

type of $(\mathcal{M}, g)$	$\mathfrak{hol}(\mathcal{M}, g)$ is contained in	
Kählerian	$\mathfrak{u}(p_0, q_0 \| p_1, q_1)$	$n = 2p_0 + 2q_0$, $m = 2p_1 + 2q_1$
special Kählerian (by def.)	$\mathfrak{su}(p_0, q_0 \| p_1, q_1)$	$n = 2p_0 + 2q_0$, $m = 2p_1 + 2q_1$
hyper-Kählerian	$\mathfrak{hosp}(p_0, q_0 \| p_1, q_1)$	$n = 4p_0 + 4q_0$, $m = 4p_1 + 4q_1$
quaternionic-Kählerian	$\mathfrak{sp}(1) \oplus \mathfrak{hosp}(p_0, q_0 \| p_1, q_1)$	$n = 4p_0 + 4q_0 \geq 8$, $m = 4p_1 + 4q_1$

$$\mathrm{Ric}(Y,Z) := \mathrm{str}\left(X \mapsto (-1)^{|X||Z|} R(Y,X)Z\right),$$

$$\mathrm{str}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \mathrm{tr}A - \mathrm{tr}D$$

Proposition. Let $(\mathcal{M}, g)$ be a Kählerian supermanifold, then $\mathrm{Ric} = 0$ if and only if $\mathfrak{hol}(\mathcal{M}, g) \subset \mathfrak{su}(p_0, q_0|p_1, q_1)$. In particular, if $(\mathcal{M}, g)$ is special Kählerian, then $\mathrm{Ric} = 0$; if M is simply connected, $(\mathcal{M}, g)$ is Kählerian and $\mathrm{Ric} = 0$, then $(\mathcal{M}, g)$ is special Kählerian.

A generalization of the Wu theorem

the product $\mathcal{M} \times \mathcal{N} = (M \times N, \mathcal{O}_{\mathcal{M}\times\mathcal{N}})$:
Let $(U, x^1, ..., x^n, \xi^1, ..., \xi^m)$ and $(V, y^1, ..., y^p, \eta^1, ..., \eta^q)$ be coordinate systems on $\mathcal{M}$ and $\mathcal{N}$
by definition, $\mathcal{O}_{\mathcal{M}\times\mathcal{N}}(U \times V) := \mathcal{O}_{M\times N}(U \times V) \otimes \Lambda_{\xi^1,...,\xi^m,\eta^1,...,\eta^q}$

a supersubalgebra $\mathfrak{g} \subset \mathfrak{osp}(p_0, q_0|2k)$ is *weakly-irreducible* if it does not preserve any non-degenerate vector supersubspace of $\mathbb{R}^{p_0+q_0} \oplus \Pi(\mathbb{R}^{2k})$.

Theorem. *Let* $(\mathcal{M}, g)$ *be a Riemannian supermanifold such that the pseudo-Riemannian manifold* $(M, \tilde{g})$ *is simply connected and geodesically complete. Then there exist Riemannian supermanifolds* $(\mathcal{M}_0, g_0), (\mathcal{M}_1, g_1), ..., (\mathcal{M}_r, g_r)$ *such that*

$$(\mathcal{M}, g) = (\mathcal{M}_0 \times \mathcal{M}_1 \times \cdots \times \mathcal{M}_r, g_0 + g_1 + \cdots + g_r), \quad (1)$$

the supermanifold $(\mathcal{M}_0, g_0)$ *is flat and the holonomy algebras of the supermanifolds* $(\mathcal{M}_1, g_1)$*,...,*$(\mathcal{M}_r, g_r)$ *are weakly-irreducible. In particular,*

$$\mathfrak{hol}(\mathcal{M}, g) = \mathfrak{hol}(\mathcal{M}_1, g_1) \oplus \cdots \oplus \mathfrak{hol}(\mathcal{M}_r, g_r).$$

For general $(\mathcal{M}, g)$ *decomposition* (1) *holds locally.*

Problem: Classify possible irreducible holonomy algebras of Riemannian supermanifolds

$\mathfrak{g} \subset \mathfrak{osp}(p,q|2m)$
The space of algebraic curvature tensors of type $\mathfrak{g}$:

$$\mathcal{R}(\mathfrak{g}) = \left\{ R \in \wedge^2(\mathbb{R}^{p,q|2m})^* \otimes \mathfrak{g} \;\middle|\; \begin{array}{c} R(X,Y)Z + (-1)^{|X|(|Y|+|Z|)} R(Y,Z)X \\ +(-1)^{|Z|(|X|+|Y|)} R(Z,X)Y = 0 \\ \text{for all homogeneous } X,Y,Z \in \mathbb{R}^{p,q|2m} \end{array} \right\}$$

$\mathfrak{g} \subset \mathfrak{osp}(p,q|2m)$ is *a Berger superalgebra* if

$$\mathrm{span}\{R(X,Y)|R \in \mathcal{R}(\mathfrak{g}),\ X,Y \in \mathbb{R}^{p,q|2m}\} = \mathfrak{g}$$

Proposition. Let $\mathcal{M}$ be a Riemannian supermanifold. Then its holonomy algebra $\mathfrak{hol}(\nabla) \subset \mathfrak{osp}(p,q|2m)$ is a Berger superalgebra.

Classification of irreducible non-symmetric Berger superalgebras $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$:

$$\begin{array}{ll} \mathfrak{osp}(p, q|2m), & \mathfrak{osp}(r|2k, \mathbb{C}), \\ \mathfrak{u}(p_0, q_0|p_1, q_1), & \mathfrak{su}(p_0, q_0|p_1, q_1), \\ \mathfrak{hosp}(r, s|k), & \mathfrak{hosp}(r, s|k) \oplus \mathfrak{sp}(1), \\ \mathfrak{osp}^{sk}(2k|r, s) \oplus \mathfrak{sl}(2, \mathbb{R}), & \mathfrak{osp}^{sk}(2k|r) \oplus \mathfrak{sl}(2, \mathbb{C}). \end{array}$$

References:

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